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A new class of induced localized coherent structures in the $(2+1)$-dimensional nonlinear Schrödinger equation

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Abstract. In this paper, we report a novel way of constructing a new class of localized coherent structures for the $(2+1)$-dimensional nonlinear Schrödinger (NLS) equation proposed by Zakharov by utilizing the freedom (arbitrary function) in the linearized version of the bilinear equation. The localized solutions for the potential are realized mainly by the interaction of the line soliton with a curved soliton. We call such solutions 'induced localized structures (induced dromions)' as the line soliton is induced by the arbitrary function present in the system.

Generating localized solutions of $(2+1)$-dimensional integrable nonlinear evolution equations (NLEEs) continues to be a challenging contemporary problem even though it has proved to be a rewarding exercise in some of the well known $(2+1)$-dimensional integrable models such as the Davey–Stewartson (DS) equation [1, 2] and the Nizhnik–Novikov–Veselov (NNV) equation [3, 4]. In both DS and NNV equations, which are the symmetric generalizations of the KdV and NLS equations respectively, the physical field is exponentially localized. On the other hand, even though the physical field of some $(2+1)$-dimensional NLEEs may not be exponentially localized, one may be able to figure out other interesting localized physical entities and this property has been exhibited by the $(2+1)$-dimensional breaking soliton equation [5], where the potential introduced to take care of the non-local term is found to be exponentially localized.

It is a known fact that a dromion solution is nothing but a two-soliton solution made out of two non-parallel ghost solitons [6] (which are visible only in the absence of the physical field). Hence, one has to look for a pair of non-parallel ghost solitons which in turn drive the boundaries present in the system. However, in certain $(2+1)$-dimensional integrable equations, the boundaries are not driven by the ghost solitons. The $(2+1)$-dimensional NLS equation proposed by Zakharov [7] and discussed recently by Strachan [8] is the best known example exhibiting this property. Thus a question arises as to how one can construct localized solutions for such equations. The answer to this question opens up the possibility of constructing a new class of localized solutions for the potential in the above-mentioned $(2+1)$-dimensional NLS equation by making use of the freedom in the associated linearized version of the bilinear equation. This unfolds a wide class of localized solutions for the $(2+1)$-dimensional NLS equation which are not identified by the Hirota method using two non-parallel ghost solitons alone. Recently, Sen-yue Lou [9] has used such a freedom in the $(2+1)$-dimensional KdV equation to construct more general dromion solutions from the basic dromion solutions reported in [4].

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We now take up the (2 + 1)-dimensional NLS equation proposed by Zakharov in the form [8]

$$\mathrm{i}q_t = q_{xy} + Vq \tag{1a}$$
$$V_x = 2\partial_y |q|^2. \tag{1b}$$

This equation has been shown to admit the Painlevé property and its soliton solutions have been derived through a bilinear formalism [10]. Unlike the DS equation, this equation does not admit ghost solitons and hence localized physical fields in the standard way. Now, under the transformation

$$q = g/\phi \tag{2a}$$
$$V = 2\partial_{xy} \log \phi \tag{2b}$$

equation (1) can be transformed into the Hirota form

$$\mathrm{i}\mathrm{D}_t g \cdot \phi = \mathrm{D}_x \mathrm{D}_y g \cdot \phi \tag{3a}$$
$$\mathrm{D}_x^2 \phi \cdot \phi = 2gg^*. \tag{3b}$$

We now expand g and ϕ in the form of a power series as

$$g = \varepsilon g^{(1)} + \varepsilon^3 g^{(3)} + \cdots \tag{4a}$$
$$\phi = 1 + \varepsilon^2 \phi^{(2)} + \varepsilon^4 \phi^{(4)} + \cdots. \tag{4b}$$

To generate the one-line-soliton solution, we substitute the above series into equation (3) and collect the resultant equations obtained by comparing various powers of ε to give

$$\varepsilon: \qquad \mathrm{i}g_t^{(1)} = g_{xy}^{(1)} \tag{5a}$$
$$\varepsilon^2: \qquad \phi_{xx}^{(2)} = g^{(1)} g^{(1)*} \tag{5b}$$
$$\varepsilon^3: \qquad \mathrm{i}g_t^{(3)} - g_{xy}^{(3)} = -(\mathrm{i}\mathrm{D}_t - \mathrm{D}_x \mathrm{D}_y) g^{(1)} \cdot \phi^{(2)} \tag{5c}$$
$$\varepsilon^4: \qquad 2\phi_{xx}^{(4)} + \mathrm{D}_x^2 \phi^{(2)} \cdot \phi^{(2)} = 2(g^{(3)} g^{(1)*} + g^{(1)} g^{(3)*}) \tag{5d}$$

and so on. Solving (5*a*), we obtain

$$g^{(1)} = \sum_{j=1}^{N} \exp(\chi_j) \qquad \chi_j = k_j x + m_j(y,t) + c_j \tag{6a}$$

where $m_j(y,t)$ is an arbitrary function of (y,t) chosen such that

$$m_j(y,t) = m_j(\rho) = m_j(y - \mathrm{i}k_j t) \tag{6b}$$

and k_j and c_j are complex constants. To construct the one-soliton solution, we take $N = 1$ and substitute $g^{(1)}$ in (5*b*) to give

$$\phi_{xx}^{(2)} = \exp(\chi_1 + \chi_1^*). \tag{7}$$

Solving the above equation, we get

$$\phi^{(2)} = \exp(\chi_1 + \chi_1^* + 2\psi) \qquad \exp(2\psi) = \frac{1}{4k_{1\mathrm{R}}^2} \qquad k_{1\mathrm{R}} = \mathrm{Re}\, k_1. \tag{8}$$

Using $g^{(1)}$ and $\phi^{(2)}$ in (5*c*) and (5*d*), one can show that $g^{(j)} = 0$ for $j \geqslant 3$ and $\phi^{(j)} = 0$ for $j \geqslant 4$. Now, using equations (2) and (6*a*) and (8), the physical field q and the potential V assume the following form

$$q = k_{1\mathrm{R}} \operatorname{sech}(\chi_{1\mathrm{R}} + \psi) \exp(\mathrm{i}\chi_{1\mathrm{I}}) \tag{9a}$$
$$V = 2k_{1\mathrm{R}} (m_{1\mathrm{R}})_{\rho_\mathrm{R}} \operatorname{sech}^2(\chi_{1\mathrm{R}} + \psi) \qquad \rho_\mathrm{R} = y + k_{1\mathrm{I}} t \qquad k_{1\mathrm{I}} = \mathrm{Im}\, k_1 \tag{9b}$$

where the function $(m_{1\mathrm{R}})_{\rho_\mathrm{R}}$ is arbitrary (as $m_1(\rho)$ is arbitrary). It is evident from the above equation (9) that both the physical field q and the potential V remain finite on the curve

$$C = \chi_{1\mathrm{R}} + \psi = k_{1\mathrm{R}}x + m_{1\mathrm{R}}(y, t) + c_{1\mathrm{R}} = 0 \tag{10}$$

and decay exponentially everywhere (as $x, y \to \infty$) apart from the curve ($C = 0$) given by equation (10). One may call such line solitons (which do not decay along a straight line) as 'curved solitons' [9].

Looking at the nature of the solutions (9*a*) and (9*b*), we see that both the physical field q and the potential V vanish when the parameter $k_{1\mathrm{R}} \to 0$. But as $(m_{1\mathrm{R}})_{\rho_\mathrm{R}} \to 0$, the potential V alone vanishes whereas the field variable q survives. This indicates that one can generate localized structures for the potential V by choosing the derivative of the arbitrary function $m_{1\mathrm{R}}$ to be localized in the ρ_R direction. In other words, by choosing $(m_{1\mathrm{R}})_{\rho_\mathrm{R}}$ to be localized in the ρ_R direction, one can easily set up the interaction of the line soliton $(m_{1\mathrm{R}})_{\rho_\mathrm{R}}$ with the curved soliton $\mathrm{sech}^2(\chi_{1\mathrm{R}} + \psi)$ so that the line and curved solitons disappear (ghosts) resulting in the formation of a singly localized structure. Thus, a single dromion is localized around the point of interaction of the curved and line solitons. For example, if one chooses

$$(m_{1\mathrm{R}})_{\rho_\mathrm{R}} = \mathrm{sech}^2(\rho_\mathrm{R}) \tag{11}$$

we have a one dromion for the potential V as

$$V = 2k_{1\mathrm{R}}\,\mathrm{sech}^2(\rho_\mathrm{R})\,\mathrm{sech}^2(k_{1\mathrm{R}}x + \tanh(\rho_\mathrm{R}) + \gamma) \qquad \rho_\mathrm{R} = y + k_{1\mathrm{I}}t \tag{12a}$$

which decays exponentially in all directions. In other words, we have induced a one-dimensional soliton at the derivative of the arbitrary function $m_{1\mathrm{R}}(\rho_\mathrm{R})$ to generate a localized solution for the potential V and hence we call them 'induced localized structures'. The dromion solution given by equation (12) is driven by a curved soliton ($C = 0$) and a line soliton ($\rho_\mathrm{R} = 0$). One can indeed replace the line soliton by a curved soliton so that the dromion can be driven by two curved solitons as the argument ρ_R can be replaced by an arbitrary function as

$$\rho_\mathrm{R} \to h(\rho_\mathrm{R}) = h. \tag{12b}$$

Thus, a $(2+1)$-dimensional dromion can be driven not only by a line soliton and a curved soliton, but also by two curved solitons. As the function $m_{j\mathrm{R}}(\rho_\mathrm{R})$ is arbitrary, one can construct even richer structures as demonstrated by Sen-yue Lou [10] for the $(2 + 1)$-dimensional KdV equation. For example, we can choose an algebraic form

$$(m_{1\mathrm{R}})_{\rho_\mathrm{R}} = \frac{1}{(\rho_\mathrm{R} + \rho_0)^2 + 1} \tag{12c}$$

to give

$$V = 2k_{1\mathrm{R}}\left[\frac{1}{(\rho_\mathrm{R} + \rho_0)^2 + 1}\right]\mathrm{sech}^2\left(k_{1\mathrm{R}}x + \int \frac{1}{(\rho_\mathrm{R} + \rho_0)^2 + 1}\,\mathrm{d}\rho_\mathrm{R} + \gamma\right). \tag{12d}$$

One can indeed generalize this procedure to construct multidromions by taking N line solitons to give

$$V_N = 2k_{1\mathrm{R}}\left(\sum_{j=1}^{N}(m_{j\mathrm{R}})_{\rho_\mathrm{R}}\right)\mathrm{sech}^2\left(k_{1\mathrm{R}}x + \sum_j m_{j\mathrm{R}}(\rho_\mathrm{R}) + \psi\right). \tag{13}$$

The above expression describes a localized structure made out of N line solitons and a curved soliton. In general, N line solitons can be chosen conveniently to decay exponentially, algebraically or in an oscillatory fashion.

From the above discussion, we infer that the first type of multidromion is being driven by a curved ghost soliton and N line solitons. Now, a question arises of whether one can find a multidromion solution driven by two or more curved ghost solitons. For example, to construct a two dromion solution, we take $N = 2$ and hence we have from equation (6*a*)

$$g^{(1)} = \exp(\chi_1) + \exp(\chi_2). \tag{14}$$

Substituting this in (5*b*), (5*c*) and (5*d*) and solving them accordingly, we obtain

$$g^{(3)} = L_1 \exp(\chi_1 + \chi_1^* + \chi_2) + L_2 \exp(\chi_2 + \chi_2^* + \chi_1) \tag{15a}$$

$$\phi^{(2)} = P_1 \exp(\chi_1 + \chi_1^*) + P_2 \exp(\chi_1 + \chi_2^*) + P_3 \exp(\chi_2 + \chi_1^*) + P_4 \exp(\chi_2 + \chi_2^*) \tag{15b}$$

$$\phi^{(4)} = P_5 \exp(\chi_1 + \chi_1^* + \chi_2 + \chi_2^*) \tag{15c}$$

where the parameters L_1, L_2, P_2 and P_3 are complex ($P_2 = P_3^*$) and P_1, P_4 and P_5 are real and they can be chosen conveniently in accordance with the equation (3). Using (14) and (15), the potential V becomes after choosing $k_1 = k_2$ (for illustration)

$$\begin{aligned}
V &= \frac{G}{F}\\
G &= 2[4k_{1R}(m_{1R})_{\rho_{1R}} P_1 \exp(\chi_1 + \chi_1^*) + 2k_{1R} P_2(m_{1\rho} + m_{2\rho}^*) \exp(\chi_1 + \chi_2^*)\\
&\quad +2k_{1R} P_3(m_{2\rho} + m_{1\rho}^*) \exp(\chi_2 + \chi_1^*) + 4k_{1R}(m_{2R})_{\rho_{1R}} P_4 \exp(\chi_2 + \chi_2^*)\\
&\quad +8k_{1R}(m_{1R} + m_{2R})_{\rho_{1R}} P_5 \exp(\chi_1 + \chi_1^* + \chi_2 + \chi_2^*) + 4k_{1R}(m_{2R})_{\rho_{1R}} P_1 P_5\\
&\quad \times \exp(2[\chi_1 + \chi_1^*] + \chi_2 + \chi_2^*) + 2k_{1R}(m_{1\rho}^* + m_{2\rho}) P_2 P_5\\
&\quad \times \exp(2\chi_1 + \chi_1^* + \chi_2 + 2\chi_2^*) + 2k_{1R}(m_{1\rho} + m_{2\rho}^*) P_3 P_5\\
&\quad \times \exp(\chi_1 + 2\chi_1^* + 2\chi_2 + \chi_2^*) + 4k_{1R}(m_{1R})_{\rho_{1R}} P_4 P_5\\
&\quad \times \exp(\chi_1 + \chi_1^* + 2[\chi_2 + \chi_2^*])]\\
F &= [1 + P_1 \exp(\chi_1 + \chi_1^*) + P_2 \exp(\chi_1 + \chi_2^*) + P_3 \exp(\chi_2 + \chi_1^*) + P_4 \exp(\chi_2 + \chi_2^*)\\
&\quad +P_5 \exp(\chi_1 + \chi_1^* + \chi_2 + \chi_2^*)]^2\\
\chi_1 &= k_1 x + m_1(\rho_1) \qquad \chi_2 = k_1 x + m_2(\rho_1).
\end{aligned} \tag{16}$$

Looking at the above solution, it is clear that it contains two arbitrary functions m_1 and m_2 and hence by properly choosing them as given by equations (11) and (12*c*), one can generate a two-dromion solution even though the solution does not have a compact form. Similarly, more generalized localized solutions can be constructed.

In this paper, by utilizing the freedom in the bilinearized version of the linear equation of the $(2+1)$-dimensional NLS equation, we have generated a new class of 'induced localized structures'. We are investigating whether the existence of such freedom in the bilinearized version of the linear equations of other $(2 + 1)$-dimensional nonlinear partial differential equations such as the sine–Gordon equation [11] can be properly harnessed to generate other kinds of localized solutions.

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